RELATIVE COMPLEXITY OF RANDOM WALKS IN RANDOM SCENERY IN THE ABSENCE OF A WEAK INVARIANCE PRINCIPLE FOR THE LOCAL TIMES

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Abstract. We answer the question of Aaronson about the relative complexity of Random Walks in Random Sceneries driven by either aperiodic two dimensional random walks, two-dimensional Simple Random walk, or by aperiodic random walks in the domain of attraction of the Cauchy distribution. A key step is proving that the range of the random walk satisfies the Fölner property almost surely.

1. Introduction

The notion of entropy was first introduced into ergodic theory by Kolmogorov as an isomorphism invariant. That is if two measure preserving systems are (measure theoretically) isomorphic then their entropy is the same. It was later shown by a seminal theorem of Ornstein [Or] that entropy is a complete invariant for Bernoulli automorphisms (transformations which are isomorphic to a shift on an i.i.d. sequence, aka Bernoulli Shifts) meaning that two Bernoulli automorphisms are isomorphic if and only if their entropies coincide. Measure theoretic complexity, which is roughly the rate of growth of information, was introduced by Ferenczi [Fe], Katok and Thouvenot [KT] and others as an isomorphism invariant for the problem of understanding whether two zero entropy systems are isomorphic.

In a recent work, Aaronson [Aa] introduced a relativised notion of complexity and calculated the relative complexity of random walks in random sceneries where the jump random variable is $\mathbb{Z}$ valued, centred, aperiodic and in the domain of attraction of an α stable distribution with $1 < \alpha \leq 2$. The main tool used there is Borodin's weak invariance principle for the local times [Bor1, Bor2]. Random Walks in Random Scenery(RWRS) are natural models for the study of this relative complexity notion as they are examples of non-Bernoulli K-automorphisms [Ka] and so the relative complexity could be a good way to try to distinguish whether two such systems are isomorphic. Indeed, Aaronson's ideas of using the weak convergence of local times to count Hamming balls were later used by Austin [Au], in the definition of a full isomorphism invariant, called the scenery entropy, for the class of random walk in random sceneries with jump distribution of finite variance.

For the purpose of the introduction, we will now describe the classical random walk in random scenery from probability theory. Let $X_1, X_2, \ldots$ be i.i.d. $\mathbb{Z}^d$-valued random variables, the jump process, and $S_n := \sum_{k=1}^{n} X_k$ the corresponding random walk. The scenery is an independent (of $\{X_i\}$) field of i.i.d random variables $\{C_j\}_{j\in\mathbb{Z}^d}$. The joint process (X_n, C_{S_n}) is then known as a random walk in random scenery. The relative complexity of Aaronson in that case is heuristically as follows: Assuming that we have full information of the sequence $\mathbf{X} = X_1, X_2, \ldots$, what is the rate of growth of the information arising from the scenery for most of the realizations of X?

If $X_1, X_2, \ldots$ are Bernoulli ± 1 fair coin tossing (in other words the driving random walk is the simple random walk on the integers) then by the local central limit theorem, at time n, the range of the random walk $R(n) := \{S_j : 1 \leq j \leq n\}$ is of order constant times $\sqrt{n}$. The range of the random walk is related to this problem since $\{C_{S_j} : j \in [1, n]\} = \{C_k : k \in R(n)\}$. It then appears that the rate of growth of information arising from the scenery should be of the order $\exp(H(C) \cdot \#R(n))$ with $H(C)$ the Shannon entropy of C. Thus for this example one would

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expect (which is verified in [Aa]) that the relative complexity is of the order $\exp(c_w\sqrt{n}H(C))$, where c_w is a constant depending on w and the order should be interpreted as existence of a non trivial distributional limit.

In this paper we treat random walks in random sceneries driven by aperiodic, recurrent, random walks with finite second moments in $\mathbb{Z}^2$, by Simple Random Walk in $\mathbb{Z}^2$, or by an aperiodic, recurrent, random walk in $\mathbb{Z}$ in the domain of attraction of the Cauchy distribution. Since the limiting distributions don't have local times, Aaronson's and Austin's methods do not apply. For these types of RWRS's, Kesten and Spitzer [KS] conjectured that there exists constants $a_n \to \infty$ such that $\frac{1}{a_n}\sum_{k=1}^{nt} Z_{S_n}$ converges weakly to a Brownian motion (when $\mathrm{var}(C) < \infty$). This was shown to be true by Bolthausen [Bol] when S_n is the two dimensional simple random walk on $\mathbb{Z}^2$ and by the first author and Utev [DU15] for the case of the Cauchy distribution. Bolthausen's argument was generalized by Černy in [Če] and the ideas there were a major inspiration for us. The idea is since one cannot have a weak invariance principle for the local time, one can study the asymptotics of self-intersection local times, see Section 4, in order to prove a statement of the form "for most of the points of $R(n)$ the number of times the random walk visits them, up to time n, is greater than a constant times $\log(n)$" (see Theorem 5 for a precise statement). We refine this method to prove a result of independent interest, namely that the range of the random walk is almost surely a Fölner sequence (Theorem 6). With these two Theorems at hand we can proceed by a simplified argument to deduce the main result, Theorem 2. which answers Aaronson's question about the relative complexity of this type of RWRS's. We think that this simpler (and softer) method can be used to calculate the relative complexity of other RWRS's such as [Ru, Ba].

This paper is organized as follows. In Section 2 we first start with the relevant definitions we need and then end with the statement of the main result. In Section 4 we prove the results we need for the random walk and it's range. Finally, Section 5 is the proof of the main Theorem. For the sake of completeness we include an Appendix with a proof of some standard facts about the random walks we consider.

2. Preliminaries

2.1. **Relative complexity over a factor.** Let $(X,\mathcal{B},m)$ be a standard probability space and $T : X \to X$ a m- preserving transformation. Denote by $\mathfrak{B}(X)$ the collection of all measurable countable partitions of X. In order to avoid confusion with notions from probability, we will denote the partitions by Greek letters $\beta \in \mathfrak{B}(X)$ and the atoms of β by $\beta^1, \beta^2, ..., \beta^{\#\beta}$, $\#\beta \in \mathbb{N}\cup\{\infty\}$. A partition β is a *generating partition* if the smallest σ-algebra containing $\{T^{-n}\beta :\ n \in \mathbb{Z}\}$ is $\mathcal{B}$. For $\beta \in \mathfrak{B}(X)$ and $n \in \mathbb{N}$ let

$$\beta_0^n := \vee_{j=0}^n T^{-j}\beta = \left\{\cap_{j=0}^n T^{-j}b_j : b_1, .., b_n \in \beta\right\}.$$

The *β-name* of a point $x \in X$ is the sequence $\beta(x) \in (\#\beta)^{\mathbb{N}}$ defined by

$$\beta_n(x) = i \text{ if and only if } T^n x \in \beta^i.$$

The (T,β,n) *Hamming pseudo-metric* on X is defined by

$$\bar{d}_n^{(\beta)}(x,y) = \frac{1}{n}\#\left\{k \in \{0,...,n-1\} :\ \beta_k(x) \neq \beta_k(y)\right\}.$$

That is two points $x, y \in X$ are $\bar{d}_n^{(\beta)}$ close if for most of the $k's$ in $\{0,..,n-1\}$, $T^k x$ and $T^k y$ lie in the same partition element of β. An ϵ-ball in the Hamming pseudo-metric will be denoted by

$$B(n,\beta,x,\epsilon) := \left\{y \in X :\ \bar{d}_n^{(\beta)}(x,y) < \epsilon\right\}.$$

This pseudo-metric was used in [KT, Fe] to define complexity sequences and slow-entropy-type invariants. It was shown for example by Katok and Thouvenot [KT] that if the growth rate of the complexity sequence is of order e^{hn} with $h > 0$, then h equals the entropy of X, by Ferenczi [Fe] that T is isomorphic to a translation of a compact group if and only if the complexity is of lesser order from any sequence which grows to infinity and more. In this paper

we will be interested with the relativised versions of these invariants which were introduced in Aaronson [Aa].

A T-invariant sub-σ-algebra $\mathcal{C}\subset\mathcal{B}$ is called a *factor*. An equivalent definition in ergodic theory is a probability preserving transformation $(Y,\tilde{\mathcal{C}},\nu,S)$ with a (measurable) factor map $\pi:X\to Y$ such that $\pi T=S\pi$ and $\nu=m\circ\pi^{-1}$, in this case $\mathcal{C}=\pi^{-1}\tilde{\mathcal{C}}$.

Given a factor $\mathcal{C}\subset\mathcal{B}$, $n\in\mathbb{N}$, $\beta\in\mathfrak{B}(X)$ and $\epsilon>0$ we define a $\mathcal{C}$-measurable random variable $\mathcal{K}_{\mathcal{C}}(\beta,n,\epsilon):X\to\mathbb{N}$ by

$$\mathcal{K}_{\mathcal{C}}(\beta,n,\epsilon)(x):=\min\left\{\#F:\ F\subset X,\ m\left(\bigcup_{z\in F}B(n,\beta,z,\epsilon)\middle|\mathcal{C}\right)(x)>1-\epsilon\right\},$$

where $m(\cdot|\mathcal{C})$ denotes the conditional measure of m with respect to $\mathcal{C}$. The sequence of random variables $\{\mathcal{K}_{\mathcal{C}}(\beta,n,\epsilon)\}_{n=1}^{\infty}$ is called the *relative complexity* of (T,β) with respect to β given $\mathcal{C}$.

The *upper entropy dimension of T given $\mathcal{C}$* is defined by

$$\overline{\mathrm{EDim}}(T,\mathcal{C}):=\inf\left\{t>0:\ \limsup_{n\to\infty}\frac{\log\mathcal{K}_{\mathcal{C}}(\beta,n,\epsilon)}{n^t}=0,\ \forall\beta\in\mathfrak{B}(X)\right\}$$

and the *lower entropy dimension of T given $\mathcal{C}$* is

$$\underline{\mathrm{Edim}}(T,\mathcal{C})=\sup\left\{t>0:\ \exists\beta\in\mathfrak{B}(X),\ \liminf_{n\to\infty}\frac{\log\mathcal{K}_{\mathcal{C}}(\beta,n,\epsilon)}{n^t}=\infty\right\}.$$

In case $\underline{\mathrm{Edim}}(T,\mathcal{C})=\overline{\mathrm{Edim}}(T,\mathcal{C})=a$ we write $\mathrm{Edim}(T,\mathcal{C})=a$ and call this quantity the *entropy dimension of T given $\mathcal{C}$*.

Given a sequence of random variables $Y_n,n\in\mathbb{N}$ taking values on $[0,\infty]$ we write $Y_n\xrightarrow[n\to\infty]{\mathfrak{D}}Y$ to denote "Y_n converges to Y in distribution" and $Y_n\xrightarrow[n\to\infty]{m}Y$ to denote convergence in probability. The next Theorem is a special case of [Aa, Thm 2] when $\{n_k\}_{k=1}^{\infty}=\mathbb{N}$.

Theorem 1 (Aaronson's Generator Theorem). *Let $(X,\mathcal{B},m,T)$ be a measure preserving transformation and a sequence $d_n>0$.*

(a) *If there is a countable T-generator $\beta\in\mathfrak{B}(X)$ and a random variable Y on $[0,\infty]$ satisfying*

$$\frac{\log\mathcal{K}_{\mathcal{C}}(\beta,n,\epsilon)}{d_n}\xrightarrow[n\to\infty,\ \epsilon\to0]{\mathfrak{D}}Y$$

Then for all T-generating partitions $\alpha\in\mathfrak{B}(X)$,

$$\frac{\log\mathcal{K}_{\mathcal{C}}(\alpha,n,\epsilon)}{d_n}\xrightarrow[n\to\infty,\ \epsilon\to0]{\mathfrak{D}}Y$$

(b) *if for some $\beta\in\mathfrak{B}(X)$, a generating partition for T,*

$$\frac{\log\mathcal{K}_{\mathcal{C}}(\beta,n,\epsilon)}{n^t}\xrightarrow[n\to\infty,\epsilon\to0]{m}0,$$

then $\overline{\mathrm{EDim}}(T,\mathcal{C})\le t$.

(c) *if for some partition $\beta\in\mathfrak{B}(X)$,*

$$\frac{\log\mathcal{K}_{\mathcal{C}}(\beta,n,\epsilon)}{n^t}\xrightarrow[n\to\infty,\epsilon\to0]{m}\infty$$

then $\underline{\mathrm{Edim}}(T,\mathcal{C})\ge t$.

2.2. **Basic ergodic theory for $\mathbb{Z}^d$ actions.** Let $(X,\mathcal{B},m)$ be a standard probability space and G be an Abelian countable group. A measure preserving action of G on $(X,\mathcal{B},m)$ is a map $\mathsf{S}:G\to\mathrm{Aut}(X,\mathcal{B},m)$ such that for every $g_1,g_2\in G$, $\mathsf{S}_{g_1g_2}=\mathsf{S}_{g_1}\mathsf{S}_{g_2}$ and for all $g\in G$, $(\mathsf{S}_g)_*m=m$. The action is *ergodic* if there are no non trivial S-invariant sets.

Given an ergodic G action $(X,\mathcal{B},m,\mathsf{S})$ and increasing sequence F_n of subsets of G one can define a sequence of averaging operators $A_n:L_2(X,\mathcal{B},m)\circlearrowleft$ by

$$A_n(f):=\frac{1}{\#F_n}\sum_{g\in F_n}f\circ\mathsf{S}_g$$

and ask whether for all $f \in L_2(X, \mathcal{B}, m)$ one has $A_n(f) \to \int_X f dm$ in L_2. The sequences of sets $\{F_n\}_{n=1}^{\infty}$ for which this is necessarily true are called *Fölner sequences* and they are characterised by the property that for every $g \in G$,

$$\frac{\#\left[F_n \triangle \{F_n + g\}\right]}{\#F_n} \xrightarrow[n\to\infty]{} 0.$$

In this work we will be concerned with either actions of $G = \mathbb{Z}$ which is generated by one measure preserving transformation or $G = \mathbb{Z}^2$ which corresponds to two commuting measure preserving transformations. For a finite partition β of X, one defines the *entropy* of S with respect to β by

$$h(\mathsf{S}, \beta) := \lim_{n\to\infty} \frac{1}{n^d} H\left(\bigvee_{j\in[0,n]^d\cap\mathbb{Z}^d} \mathsf{S}_j^{-1}\beta\right),$$

where $H(\beta) = \sum_{j=1}^{\#\beta} m\left(\beta^i\right) \log m\left(\beta^i\right)$ is the *Shannon entropy* of the partition. The entropy of S is then defined by

$$h(\mathsf{S}) = \sup_{\beta\in\mathfrak{B}(X):\ \beta \text{ finite}} h(\mathsf{S}, \beta)$$

As in the case of a $\mathbb{Z}$ action, one says that β is a generating partition if the smallest sigma algebra containing $\vee_{j\in\mathbb{Z}^d} \mathsf{S}_j^{-1}\beta$ is $\mathcal{B}$. In an analogous way to the case of $\mathbb{Z}$ actions, it follows that if β is a generating partition for S then $h(\mathsf{S}) = h(\mathsf{S}, \beta)$ and if $h(\mathsf{S}) < \infty$ then there exists finite generating partitions [Kr, KW, DP].

3. Random walks in random sceneries and statement of main theorem

In what follows we will be interested in a random walk in random scenery where the jump random variable $\xi \in \mathbb{Z}^2$ is in the domain of attraction of 2-dimensional Brownian Motion or $\xi \in \mathbb{Z}$ is strongly aperiodic and in the domain of attraction of the Cauchy law. The reason that these two models are of most interest to us is that the limiting distribution does not have a local time process.

To be more precise let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. $\mathbb{Z}^d$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with characteristic function $\phi_\xi(t) := \mathbb{E}(e^{it\cdot\xi})$ for $t \in [-\pi, \pi]^d$, and that either

A1: (1-*stable*) $\xi \in \mathbb{Z}$ and $\phi_\xi(t) = 1 - \gamma|t| + o(|t|)$ for $t \in [-\pi, \pi]$, for some $\gamma > 0$; or

A2: ξ is in $\mathbb{Z}^2$ and $\mathbb{E}\,|\xi|^2 < \infty$ with non-singular covariance matrix Σ; equivalently $\phi_\xi(t) = 1 - \langle t, \Sigma t\rangle + o(|t|^2)$ for $t \in [-\pi, \pi]^2$.

In the above cases the random walk $S_n(\xi) := \xi_1 + \xi_2 + \cdots + \xi_n$ we will also assume that the random walk is *strongly aperiodic* in the sense that there is no proper subgroup L of $\mathbb{Z}^d$ such that $\mathbb{P}(\xi - x \in L) = 1$ for some $x \in \mathbb{Z}^d$.

We are also interested in the two dimensional *Simple Random Walk*, which has period 2 and is thus not covered by **A2** above.

A2': $\xi \in \mathbb{Z}^2$ and $\mathbb{P}[\xi = e] = 1/4$ for $|e| = 1$. Then $\sqrt{\det(\Sigma)} = 1/2$.

Denote by μ_ξ the distribution of ξ. The base of the RWRS is then defined as $\Omega = \left(\mathbb{Z}^d\right)^{\mathbb{N}}$ the space of all $\mathbb{Z}^d$-valued sequences, $\mathbb{P} = \prod_{k=1}^{\infty} \mu_\xi$, the product measure, and $\sigma : \Omega \to \Omega$ the left shift on Ω defined by

$$(\sigma w)_n = w_{n+1}.$$

When $d = 2$, the random scenery is an ergodic probability preserving $\mathbb{Z}^2$- action $(Y, \mathcal{C}, \nu, \mathsf{S})$ and when $d = 1$ it is just an ergodic probability preserving transformation $\mathsf{S} : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$.

The skew product transformation on $Z = \Omega \times Y$, $\mathcal{B}_Z = \mathcal{B}_\Omega \otimes \mathcal{B}_Y$, $m = \mathbb{P} \times \nu$, defined by

$$T(w, y) = (\sigma w, \mathsf{S}_{w_1}(y)),$$

is the *random walk in random scenery* with scenery $(Y, \mathcal{C}, \nu, \mathsf{S})$ and *jump random variable* ξ.

Theorem 2. *Let $(Z, \mathcal{B}_Z, m, T)$ be RWRS with random scenery $(Y, \mathcal{C}, \nu, \mathsf{S})$ and jump random variable ξ.*

(a) *If $d=1$ and ξ satisfies **A1** then for any generating partition β for T,*

$$\frac{\log(n)}{\pi\gamma n}\mathcal{K}_{\mathcal{B}_\Omega}(\beta,n,\epsilon)\xrightarrow{m} h(\mathsf{S}).$$

(b) *If $d=2$ and ξ satisfies **A2** or **A2'** then for any generating partition β for T,*

$$\frac{\log n}{2\pi\sqrt{\det(\Sigma)}n}\mathcal{K}_{\mathcal{B}_\Omega}(\beta,n,\epsilon)\xrightarrow{m} h(\mathsf{S}).$$

In particular in both cases

$$\operatorname{Edim}(T,\mathcal{B}_\Omega)=1.$$

Remark 3. *This Theorem states that the rate of growth of the complexity is of the order $\#R(n)$ where $R(n)$ is the range of the random walk up to time n. This conclusion is similar to the conclusion of Aaronson for the case where the random walk is in the domain of attraction of an α-stable random variable with $1<\alpha\leq 2$. Our method of proof can apply to these cases as well. In addition since we are not using the full theory of weak convergence of local times one can hope that this method will apply also to a wider class of dependent jump distributions.*

Two probability preserving transformations $(X_i,\mathcal{B}_i,m_i,T_i), 1=1,2$, are relatively isomorphic over the factors $\mathcal{C}_i\subset\mathcal{B}_i$ if there exists a measurable isomorphism $\pi:(X_1,\mathcal{B}_1,m_1,T_1)\to(X_2,\mathcal{B}_2,m_2,T_2)$ such that $\pi^{-1}\mathcal{C}_2=\mathcal{C}_1$. The following corollary follows from Theorem 2 together with [Aa, Corollary 4].

Corollary 4. *Suppose that $(Z_i,\mathcal{B}_{Z_i},m_i,T_i)$, $i=1,2$, are two Random walks in random sceneries with strongly aperiodic $\mathbb{Z}^2$ valued jump random variable ξ which satisfy **A2** and their sceneries $\mathsf{S}^{(i)}$ have finite entropies.*
If these two systems are isomorphic over their bases $\mathcal{B}_{\Omega_i}$ then

$$\sqrt{\det(\Sigma_1)}h(\mathsf{S}^{(1)})=\sqrt{\det(\Sigma_2)}h(\mathsf{S}^{(2)}).$$

4. The range of the random walk

Let $R(n)=\{S(1),\cdots,S(n)\}$, be the range of the random walk and for $x\in\mathbb{Z}^d$ define the local time,

$$l(n,x)=\sum_{j=1}^{n}\mathbf{1}\{S(j)=x\}.$$

Denote by $\mathcal{F}$ the σ-algebra generated by $\{X_n\}_{n=1}^\infty$

The following theorem extends [Če, Theorem 2].

Theorem 5. *Let Y_n be a point chosen uniformly at random from $R(n)$, that is*

$$\mathbb{P}[Y_n=x\mid\mathcal{F}]=\frac{\mathbb{1}\{x\in R(n)\}}{\#R(n)}. \tag{1}$$

(i) *If **A1** holds, then*

$$\mathbb{P}\Big[\pi\gamma\frac{l(n,Y_n)}{\log n}\geq u\Big|\mathcal{F}\Big]\to \mathrm{e}^{-u},\quad \text{a.s. as } n\to\infty; \tag{2}$$

(ii) *If **A2**([Če, Theorem 2]) or **A2'** holds then*

$$\mathbb{P}\Big[2\pi\sqrt{\det(\Sigma)}\frac{l(n,Y_n)}{\log n}\geq u\Big|\mathcal{F}\Big]\to \mathrm{e}^{-u},\quad \text{a.s. as } n\to\infty; \tag{3}$$

The following is the main result of this section.

Theorem 6. *Suppose that **A1**, **A2** or **A2'** holds, then $R(n)$ is almost surely a Fölner sequence, that is for all $w\in\mathbb{Z}^d$*

$$\frac{\#\Big[R(n)\triangle(R(n)+w)\Big]}{\#R(n)}\to 0. \tag{4}$$

4.1. **Proof of Theorem 5.** First of all, recall that the result under **A2** has been proven in [Če]. We will therefore focus on the remaining cases. We write C for a generic positive constant.

4.1.1. *Auxiliary results.* Before we embark on the proof of Theorem 5, we require several standard results.

The next result is a direct consequence of strong aperiodicity and Assumptions **A1** and **A2**. Its proof is a standard application of Fourier inversion, and is included in the Appendix for the sake of completeness.

Lemma 7. *Suppose that **A1** or **A2** holds. Then with* $\gamma_1 := \pi\gamma$ *and* $\gamma_2 := 2\pi\sqrt{|\Sigma|}$

$$\sup_w \mathbb{P}[S(m) = w] = O\left(\frac{1}{m}\right), \tag{5}$$

$$\mathbb{P}[S(m) = w] - \mathbb{P}[S(m) = 0] = O\left(\frac{|w|}{m^2}\right) \tag{6}$$

$$\mathbb{P}[S(m) = w] \sim \frac{1}{\gamma_d m}. \tag{7}$$

Lemma 8. *Suppose that **A1** holds. Then as* $\lambda \uparrow 1$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda\phi(t)} \sim \frac{1}{\pi\gamma}\log\Big(\frac{1}{1-\lambda}\Big). \tag{8}$$

Since simple random walk is not aperiodic, to prove Theorem 5 for the case **A2'** we recall the following (see [Law, Theorem 1.2.1]).

Lemma 9. *Under* **A2'**

$$\sup_x \mathbb{P}[S(m) = x] = O\Big(\frac{1}{m}\Big), \tag{9}$$

$$\sum_{k=0}^{n} \mathbb{P}[S_m = 0] \sim \frac{1}{\pi}\log n \tag{10}$$

For $\alpha \geq 0$ we define the *α-fold self-intersection local time*

$$L_n(\alpha) := \sum_{x\in\mathbb{Z}^d} l(n,x)^\alpha, \qquad \alpha > 0$$

$$L_n(0) := \lim_{\alpha\downarrow 0} L_n(\alpha) = \sum_{x\in\mathbb{Z}^d} \mathbf{I}\{l(n,x) > 0\} = \#R(n).$$

We will need the following strong law of large numbers which is given in [Če] for the case **A2**. The case **A2'** is included in order to demonstrate how one can handle the periodic case.

Proposition 10. *For $d = 1, 2$, and any integer $k \geq 1$ if* **A1** *or* **A2'** *holds then as $n \to \infty$*

$$\mathbb{E}\, L_n(k) \sim \frac{\Gamma(k+1)}{(\pi\gamma_d)^{k-1}} n(\log n)^{k-1}, \tag{11}$$

$$\mathrm{var}(L_n(k)) = O(n^2(\log n)^{2k-4}), \tag{12}$$

$$\lim_{n\to\infty} \frac{n(\log n)^{k-1}}{(\pi\gamma_d)^{k-1}} L_n(k) = \Gamma(k+1), \qquad \textit{almost surely.} \tag{13}$$

Proof of Proposition 10. Once (11) and (12) have been established, (13) follows for geometric subsequences by Chebyshev's inequality, and the complete result by the same argument as in Černy [Če].

Case **A1***:* The estimate (12) is contained in Theorem 3 of Deligiannidis and Utev [DU15]. It remains to prove (11).

Similar to [Če], we write

$$\mathbb{E}\, L_n(k) = \sum_{j_1,\dots,j_k=0}^{n} \mathbb{P}[S_{k_1} = \cdots = S_{j_k}] = \sum_{b=1}^{k} \rho(b,k) \sum_{0\le j_1<\cdots<j_b\le n} \mathbb{P}[S_{j_1} = \cdots = S_{j_b}],$$

where $\rho(k,k) = k!$, while the remaining factors will not be important.

Letting

$$M_n(b) := \{(m_0,\dots,m_b) \in \mathbb{N}^{b+1} : m_1,\dots,m_{b-1} \ge 1, \sum m_i = n\}, \tag{14}$$

we have by the Markov property

$$a_b(n) := \sum_{0\le j_1<\cdots<j_b\le n} \mathbb{P}[S_{j_1} = \cdots = S_{j_b}] = \sum_{m\in M_n(b)} \prod_{i=1}^{b-1} \mathbb{P}[S_{m_i} = 0]. \tag{15}$$

Then for $\lambda \in [0,1)$, by standard Fourier inversion

$$\begin{aligned}
\sum_{n=0}^{\infty} a_b(n)\lambda^n &= \sum_{n=0} \lambda^n \sum_{m\in M_n(b)} \prod_{i=1}^{b-1} \mathbb{P}[S_{m_i} = 0] \\
&= \sum_{m_0\ge 0} \sum_{m_1,\dots,m_{b-1}\ge 1}^{\infty} \sum_{n=0}^{\infty} \lambda^{m_0+\cdots+m_{b-1}+n} \prod_{i=1}^{b-1} \mathbb{P}[S_{m_i} = 0] \\
&= \sum_{m_0=0}^{\infty} \lambda^{m_0} \sum_{n=0}^{\infty} \lambda^n \prod_{i=1}^{b-1} \sum_{m_i=1}^{\infty} \lambda^{m_i}\, \mathbb{P}[S_{m_i} = 0] \\
&= \frac{1}{(1-\lambda)^2} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda\phi(t)} \right]^{b-1} \sim \frac{(\pi\gamma)^{1-b}}{(1-\lambda)^2} \log\left(\frac{1}{1-\lambda}\right)^{b-1},
\end{aligned}$$

as $\lambda \uparrow 1$, by Lemma 8. Then under **A1** (11) follows by Karamata's Tauberian theorem, since the sequence $a_b(n)$ is monotone increasing.

Case **A2'***:* The estimate (11) follows from (15) and (10).

The proof of (12) can be adapted from [DU15]. The variance is given by

$$\begin{aligned}
\mathrm{var}(L_n(k)) = C(k) \sum_{i_1\le\cdots\le i_k} \sum_{l_1\le\cdots\le l_k} \Big\{ & \mathbb{P}\left[S(i_1) = \cdots = S(i_k); S(l_1) = \cdots = S(l_k)\right] \\
& - \mathbb{P}\left[S(i_1) = \cdots = S(i_k)\right] \mathbb{P}\left[S(l_1) = \cdots = S(l_k)\right]
\end{aligned}$$

The terms where $l_1,\dots,l_k$ is not completely contained in any of the intervals $[i_j, i_{j+1}]$ can be bounded above by the positive term in the sum using (9) and the approach in [DU15]. A similar, albeit more involved, calculation is performed in the proof of Proposition 12.

Suppose then that $l_1,\dots,l_k \in [i_j, i_{j+1}]$ for some j, and by symmetry we can take $j = 1$. Define $M_n(2k)$ as in (14) and change variables to

$$\begin{gathered}
i_1 = m_0,\ \ l_1 = m_0 + m_1,\ \ l_2 = m_0 + m_1 + m_2, \dots,\ l_k = m_0 + \cdots + m_k \\
i_2 = l_k + m_{k+1},\ \ \cdots, i_k = l_k + m_{k+1} + \cdots + m_{2k-1}.
\end{gathered}$$

Write $p(m) = \mathbb{P}[S(m) = 0]$ and $\bar{p}(m) = 1/(\pi m)$. The contribution of these terms is then

$$J_n(k) = C(k) \sum_{M_n(2k)} \prod_{\substack{1\le j\le 2k-1 \\ j\ne 1,k+1}} p(m_j) \times \Big\{ p(m_1 + m_{k+1}) - p(m_1 + \cdots + m_{k+1}) \Big\}.$$

By [LL, Theorem 2.1.3] we have that

$$|p(m) + p(m+1) - 2\bar{p}(m)| \le \frac{C}{m^2}.$$

Let $q := m_2 + \cdots + m_k$ and

$$M := n - \sum_{\substack{0\le j\le 2k-1 \\ j\ne 1,k+1}} m_j.$$

Then

$$
\begin{aligned}
&\Big|\sum_{m_1+m_{k+1}=0}^{M} p(m_1+m_{k+1}) - p(m_1+m_{k+1}+q)\Big| \\
&\le \sum_{m_1=0}^{M}\sum_{m_{k+1}=0}^{[(M-m_1)/2]} \Big|p(m_1+2m_{k+1}) + p(m_1+2m_{k+1}+1) \\
&\qquad - p(m_1+2m_{k+1}+q) - p(m_1+2m_{k+1}+1+q)\Big| \\
&\le \sum_{m_1=0}^{M}\sum_{m_{k+1}=0}^{[(M-m_1)/2]} \Big\{|\bar{p}(m_1+2m_{k+1}) - \bar{p}(m_1+2m_{k+1}+q)| + \frac{C}{(m_1+2m_{k+1})^2}\Big\} \\
&\le \sum_{m_1=0}^{M}\sum_{m_{k+1}=0}^{[(M-m_1)/2]} \Big\{\frac{q}{(m_1+2m_{k+1})(m_1+2m_{k+1}+q)} + \frac{C}{(m_1+2m_{k+1})^2}\Big\} \\
&\le \sum_{m_1+m_{k+1}=0}^{n} \Big\{\frac{q}{(m_1+m_{k+1})(m_1+m_{k+1}+q)} + \frac{C}{(m_1+m_{k+1})^2}\Big\}.
\end{aligned}
$$

Thus going back to $J_n(k)$ we have

$$
\begin{aligned}
J_n(k) &\le \sum_{M_n(2k)} \prod_{\substack{1\le j\le 2k-1\\ k\ne 1,k+1}} p(m_k) \times \Big\{\frac{m_2+\cdots+m_k}{(m_1+\cdots+m_{k+1})(m_1+m_{k+1})} + \frac{C}{(m_1+m_{k+1})^2}\Big\} \\
&= \sum_{M_n(2k)} \prod_{\substack{1\le k\le 2k-1\\ k\ne 1,k+1}} p(m_k)\frac{m_2+\cdots+m_k}{(m_1+\cdots+m_{k+1})(m_1+m_{k+1})} + O\big(n(\log n)^{2k-2}\big) \\
&=: J'_n(k) + O\big(n(\log n)^{2k-2}\big).
\end{aligned}
$$

By symmetry after we split the sum in the numerator and we combine $m = m_1 + m_{k+1}$

$$
\begin{aligned}
J'_n(k) &\le Ckn \sum_{m_1,\dots,m_{2k-1}=0}^{n} \frac{1}{m_3\cdots m_{2k-1}(m_1+m_{k+1})(m_1+\cdots+m_{k+1})} \\
&\le Cn(\log n)^{2k-4} \sum_{m,m_2=0}^{n} \frac{1}{m+m_2} \le Cn^2(\log n)^{2k-4}. \qquad \square
\end{aligned}
$$

Remark 11. *A similar proof can be performed for any periodic random walk, by summing over the period.*

4.1.2. *Proof of Theorem 5.*

Proof of Theorem 5. The proof is very similar to [Če] and the end of Theorem 6 and is thus ommitted. We just point out that under **A1**

$$
\frac{\log(n)}{\pi\gamma n}\#R(n) \to 1, \qquad \text{a.s. as } n\to\infty, \tag{16}
$$

by a simple application of Result 2 in Le Gall and Rosen [RL] with $\beta = d = 1$ and $s(n)\equiv 1$, after one notices that in our case the truncated *Green's function* satisfies

$$
h(n) := \sum_{k=0}^{n} \mathbb{P}(S_k = 0) \sim \frac{\log(n)}{\pi\gamma}, \tag{17}
$$

by Lemma 8 and Karamata's Tauberian theorem.

For **A2'** note that [DE, Theorem 4] states that

$$
\frac{\log n}{\pi n}\#R(n) \to 1. \qquad \square
$$

4.2. **Proof of the Fölner property of the Range (Theorem 6).** Let $\alpha > 0$ and define

$$L_{n,w}(\alpha) := \sum_{x \in \mathbb{Z}^d} l(n,x)^\alpha l(n,x+w)^\alpha.$$

These quantities are of interest since

$$\begin{aligned} L_{n,w}(0) &:= \lim_{\alpha \downarrow 0} L_{n,w}(\alpha) = \sum_{x \in \mathbb{Z}^d} \mathbf{I}\Big(l(n,x) > 0\Big)\mathbf{I}\Big(l(n,x+w) > 0\Big) \\ &= \#(R(n) \cap R(n) + w). \end{aligned}$$

Using the above notation the Fölner property (4) can be written as

$$\lim_{n \to \infty} \frac{L_{n,w}(0)}{L_{n,0}(0)} = 1.$$

We will use the following result.

Proposition 12. *Assume* **A1** *or* **A2** *holds. For all* $w \in \mathbb{Z}^d$ *and* $\alpha \in \mathbb{Z}$, $\alpha \geq 1$

$$\frac{L_{n,w}(\alpha)}{n(\log n)^{2\alpha-1}} \to \begin{cases} \frac{\Gamma(2\alpha+1)}{(\pi\gamma)^{2\alpha-1}}, & \text{for } d = 1 \\ \frac{\Gamma(2\alpha+1)}{(2\pi\sqrt{|\Sigma|})^{2\alpha-1}}, & \text{for } d = 2. \end{cases}$$

We first complete the proof of Theorem 6 and then we will prove the above Proposition.

Proof of Theorem 6. We first treat the cases **A1** and **A2**.

Let Y_n be defined as in (1). Setting $\gamma_d = 2\pi\sqrt{|\Sigma|}$ for $d = 2$ and $\gamma_d = \pi\gamma$ for $d = 1$, define

$$Z_n := \gamma_d^2 \frac{l(n, Y_n) l(n, Y_n + w)}{\log(n)^2}.$$

For integer α, by Proposition 12

$$\begin{aligned} \mathbb{E}[Z_n^\alpha | \mathcal{F}] &= \frac{\gamma_d^{2\alpha}}{\#R(n)} \sum_x \frac{l(n,x)^\alpha l(n,x+w)^\alpha}{\log(n)^{2\alpha}} \\ &= \frac{\gamma_d^{2\alpha-1} L_{n,w}(\alpha)}{n \log(n)^{2\alpha-1}} \frac{\gamma_d n / \log(n)}{R(n)} \to \Gamma(2\alpha+1). \end{aligned}$$

These are the moments of Y^2, where $Y \sim \mathrm{Exp}(1)$. Since

$$\limsup_{k \to \infty} \frac{\Gamma(1+2k)^{1/2k}}{2k} = \lim \frac{\Gamma(1+2k)^{1/2k}}{2k} = \mathrm{e}^{-1} < \infty,$$

these moments define a unique distribution on the positive real line (see [D]), and therefore $\mathbb{P}$-almost surely, we have that conditionally on $\mathcal{F}$, $Z_n \to Y^2$ in distribution. Then

$$\begin{aligned} \frac{\sum_x \mathbf{I}(l(n,x) > 0, l(n,x+w) > 0)}{\#R(n)} &= \lim_{\alpha \downarrow 0} \frac{\gamma_d^{2\alpha}}{R(n)} \sum_x \frac{l(n,x)^\alpha l(n,x+w)^\alpha}{\log(n)^{2\alpha}} \\ &= \lim_{\alpha \downarrow 0} \mathbb{E}[Z_n^\alpha | \mathcal{F}], \quad \text{and by monotone convergence} \\ &= \mathbb{E}\Big[\lim_{\alpha \to 0} Z_n^\alpha \Big| \mathcal{F}\Big] = \mathbb{P}(Z_n > 0 | \mathcal{F}). \end{aligned}$$

This shows that

$$\frac{\sum_x \mathbf{I}(l(n,x) > 0, l(n,x+w) > 0)}{\#R(n)} = \mathbb{P}(Z_n > 0 | \mathcal{F}) \xrightarrow[n \to \infty]{} \mathbb{P}(Y^2 > 0) = 1.$$

Simple Random Walk. For the simple random walk in $\mathbb{Z}^2$ notice that one can consider the *lazy* version of the random walk, where $\mathbb{P}[\xi' = 0] = 1/2$ while for $e \in \mathbb{Z}^2$, with $|e| = 1$ we have $\mathbb{P}[\xi' = e] = 1/4d$. Then the lazy simple random walk $S_n' := \sum_{i=1}^n \xi'$, is strongly aperiodic and

satisfies **A2'** and therefore letting $R'(n) := \{S'(0), \dots, S'(n)\}$ be the range of $\{S'(n)\}_n$ we have for all $w \in \mathbb{Z}^2$

$$\frac{\#(R'(n) \cap R'(n) + w)}{\#R'(n)} \to 1,$$

almost surely. Define recursively the successive jump times

$$T_0 := \min\{j \geq 1 : S'_j \neq S'_{j-1}\}, \quad T_k := \min\{j > T_{k-1} : S'_j \neq S'_{j-1}\}.$$

Notice that the range of the simple random walk $R(n)$ is equal to the range of the lazy walk at the time of the n-th jump, $R'(T_n)$. Therefore

$$\frac{\#(R(n) \cap R(n) + w)}{\#R(n)} = \frac{\#(R'(T_n) \cap R'(T_n) + w)}{\#R'(T_n)} \to 1,$$

since $T_n \to \infty$ almost surely. □

Remark 13. *Note that it is also possible to prove Theorem 6 under* **A2'** *directly, by proving the corresponding version of Proposition 12 and then following the same argument as for* **A2**. *To adapt the variance calculation in Proposition 12 to the simple random walk, one has to sum first over the period similarly to the proof of Proposition 10.*

Proof of Proposition 12. First we prove the result for $\alpha \in \mathbb{N}$ and then we extend it to the general case $\alpha \geq 0$. For $\alpha \in \mathbb{N}$, we have

$$\begin{aligned} L_{n,w}(\alpha) &= \sum_{x \in \mathbb{Z}^2} \Big(\sum_{i=0}^{n} \mathbf{I}(S_i = x) \Big)^{\alpha} \Big(\sum_{i=0}^{n} \mathbf{I}(S_i = x + w) \Big)^{\alpha} \\ &= \sum_{x \in \mathbb{Z}^2} \sum_{i_1, \cdots, i_\alpha = 0}^{n} \mathbf{I}\Big[S(i_1) = \cdots = S(i_\alpha) = x\Big] \sum_{k_1, \cdots, k_\alpha = 0}^{n} \mathbf{I}\Big[S(k_1) = \cdots = S(k_\alpha) = x + w\Big] \\ &= \sum_{i_1, \cdots, i_{2\alpha} = 0}^{n} \mathbf{I}\Big\{S(i_1) = \cdots = S(i_\alpha) = S(i_{\alpha+1}) - w = \cdots = S(i_{2\alpha}) - w\Big\}, \end{aligned}$$

which for $w = 0$ is corresponds to the term $L_n(2\alpha)$. Then we can rewrite $L_{n,w}(\alpha)$ as

$$\begin{aligned} L_{n,w}(\alpha) = \sum_{\beta=1}^{2\alpha} \sum_{j=(\beta-\alpha)\vee 0}^{\alpha \wedge \beta} \sum_{\boldsymbol{\epsilon} \in E(\beta, j)} & j!(\beta - j)! \\ & \sum_{0 \leq i_1 < \cdots < i_\beta \leq n} \mathbf{I}\Big\{S(i_1) + \epsilon_1 w = \cdots = \cdots = S(i_\beta) + \epsilon_\beta w\Big\}, \end{aligned} \tag{18}$$

where the third sum is over the set

$$E(\beta, j) := \{\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_\beta) \in \{-1, 0\}^\beta : \sum |\epsilon_i| = j\}.$$

Expectation of $L_{n,w}(\alpha)$. For given β, n and $\boldsymbol{\epsilon} \in E(\beta, j)$, we have using the Markov property

$$\begin{aligned} \alpha(\boldsymbol{\epsilon}, \beta, n) &:= \mathbb{E} \sum_{0 \leq i_1 < \cdots < i_\beta \leq n} \mathbf{I}\Big\{S(i_1) + \epsilon_1 w = \cdots = \cdots = S(i_\beta) + \epsilon_\beta w\Big\} \\ &= \sum_{m \in M_n(\beta)} \prod_{i=1}^{\beta-1} \mathbb{P}[S(m_i) = (\epsilon_i - \epsilon_{i+1}) w], \end{aligned}$$

Next we show that the asymptotic behaviour does not actually depend on w or $\boldsymbol{\epsilon}$. In this direction we rewrite

$$\begin{aligned} \alpha(\boldsymbol{\epsilon}, \beta, n) &= \sum_{m \in M_n} \prod_{i=1}^{\beta-1} \mathbb{P}[S(m_i) = 0] + \sum_{m \in M_n} \Big\{ \prod_{i=1}^{\beta-1} \mathbb{P}[S(m_i) = (\epsilon_i - \epsilon_{i+1}) w] - \prod_{i=1}^{\beta-1} \mathbb{P}[S(m_i) = 0] \Big\} \\ &=: \alpha(\mathbf{0}, \beta, n) + \mathcal{E}(\boldsymbol{\epsilon}, \beta, n, w), \end{aligned}$$

and we claim that $\mathcal{E}(\beta, n, w) = o(\alpha(\mathbf{0}, \beta, n))$ as $n \to \infty$.

Letting $\delta_i = \epsilon_i - \epsilon_{i+1}$ we telescope the product to get

$$\mathcal{E}(\boldsymbol{\epsilon}, \beta, n, w) = \sum_{m \in M_n} \prod_{i=1}^{\beta-1} \mathbb{P}[S_{m_i} = \delta_i w] - \prod_{i=1}^{\beta-1} \mathbb{P}[S_{m_i} = 0] \tag{19}$$

$$= \sum_{j=0}^{\beta-1} \sum_{m \in M_n} \prod_{i=1}^{\beta-1-j} \mathbb{P}[S(m_i) = \delta_i w] \times \Big[\mathbb{P}[S(m_{\beta-1-j+1}) = \delta_{\beta-1-j+1} w] - \mathbb{P}[S(m_{\beta-1-j+1}) = 0] \Big]$$

$$\times \prod_{l=\beta-1-j+2}^{\beta-1} \mathbb{P}[S(m_l) = 0],$$

where implicitly the indices are not allowed to exceed their corresponding ranges.

We analyse the first term in detail

$$\Big| \sum_{m \in M_n} \prod_{i=1}^{\beta-2} \mathbb{P}[S(m_i) = \delta_i w] \times \Big[\mathbb{P}[S(m_{\beta-1}) = \delta_{\beta-1} w] - \mathbb{P}[S(m_{\beta-1}) = 0] \Big] \Big|$$

$$\leq \sum_{m_0, \ldots, m_{\beta-2}=0}^{n} \prod_{i=1}^{\beta-2} \mathbb{P}[S(m_i) = \delta_i w] \times \sum_{m_{\beta-1}=0}^{n} \Big| \mathbb{P}[S(m_{\beta-1}) = \delta_{\beta-1} w] - \mathbb{P}[S(m_{\beta-1}) = 0] \Big|$$

$$\leq \sum_{m_0, \ldots, m_{\beta-2} 0}^{n} \prod_{i=1}^{\beta-2} \mathbb{P}[S(m_i) = \delta_i w] \Big[1 + \sum_{m_{\beta-1}=1}^{\infty} \frac{C}{m^2} \Big] = O(n \log(n)^{\beta-2}),$$

by Lemma 7, the remaining errors being very similar.

The asymptotic behaviour of $\alpha(\mathbf{0}, \beta, n)$ follows from [Če] for $d = 2$ and Lemma 8 for $d = 1$ and is given by

$$\alpha(\mathbf{0}, \beta, n) \sim n \Big(\frac{\log n}{\gamma_d} \Big)^{\beta-1},$$

where $\gamma_1 := \pi\gamma$ and $\gamma_2 := 2\pi\sqrt{\det(\Sigma)}$. Going back to (18) we see that the leading term is for $\beta = 2\alpha$, and from the above discussion, we can replace all terms $\alpha(\boldsymbol{\epsilon}, 2\alpha, n)$ by $\alpha(\mathbf{0}, 2\alpha, n)$. Since $\#E(2\alpha, \alpha)(\alpha!)^2 = \Gamma(2\alpha + 1)$ we conclude that

$$\mathbb{E}\, L_{n,w}(\alpha) = \mathbb{E} \sum_{x \in \mathbb{Z}^d} l(n, x)^\alpha l(n, x + w)^\alpha \sim \Gamma(2\alpha + 1)\, n \Big(\frac{\log n}{\gamma_d} \Big)^{2\alpha-1}.$$

Variance of $L_{n,w}(\alpha)$. To compute the variance we will follow the approach developed in [DU15]. First notice that

$$\mathbb{E}\, L_{n,w}(\alpha)^2 = \mathbb{E} \sum_{i_1, \ldots, i_{2\alpha}=0}^{n} \mathbf{I}\Big(S(i_1) = \cdots = S(i_\alpha) = S(i_{\alpha+1}) - w = \cdots = S(i_{2\alpha}) - w \Big)$$

$$\times \sum_{j_1, \ldots, j_{2\alpha}=0}^{n} \mathbf{I}\Big(S(j_1) = \cdots = S(j_\alpha) = S(j_{\alpha+1}) - w = \cdots = S(j_{2\alpha}) - w \Big)$$

Let A_m, A'_m be 0 or 1 according to whether there is a w or not in the m-th increment. Then

$$\begin{aligned}\mathrm{var}\,(L_{n,w}(\alpha)) = \sum_{k_1, \ldots, k_{2\alpha}} \sum_{l_1, \ldots, l_{2\alpha}} \Big\{ \mathbb{P}\Big[& S(k_1) = S(k_2) + A_2 w = \cdots = S(k_{2\alpha}) + A_{2\alpha} w; \\ & S(l_1) = S(l_2) + A'_2 w = \cdots = S(l_{2\alpha}) + A'_{2\alpha} w \Big] \\ & - \mathbb{P}\Big[S(k_1) = S(k_2) + A_2 w = \cdots = S(k_{2\alpha}) + A_{2\alpha} w \Big] \\ & \times \mathbb{P}\Big[S(l_1) = S(l_2) + A'_2 w = \cdots = S(l_{2\alpha}) + A'_{2\alpha} w \Big] \Big\}.\end{aligned}$$

As we shall see the presence of w does not affect the asymptotic. The main role is played by the interlacement of the sequences $\mathbf{k} = (k_1, \ldots, k_{2\alpha})$ and $\mathbf{l} = (l_1, \ldots, l_{2\alpha})$. In order to define the

interlacement index $v(\mathbf{k},\mathbf{l})$, of two sequences $\mathbf{k} = (k_1,\dots,k_r)$ and $\mathbf{l} = (l_1,\dots,l_s)$, let $\mathbf{j}$ be the combined sequence of length $r+s$, where ties between elements of $\mathbf{k}$ and $\mathbf{l}$ are counted twice. We also define $\boldsymbol{\epsilon} = (\epsilon_1,\dots,\epsilon_{r+s})$, where $\epsilon_i = 1$ if the i-th element of the combined sequence is from $\mathbf{k}$ and 0 if it is from $\mathbf{l}$; that is $j_i \in \mathbf{k}$ and 0 otherwise. Then we define the *interlacement index*,

$$v(\mathbf{k},\mathbf{l}) = v(k_1,\dots,k_r;l_1,\dots,l_s) := \sum_{i=1}^{r+s-1} |\epsilon_{i+1} - \epsilon_i|, \tag{20}$$

which counts the number of times $\mathbf{k}$ and $\mathbf{l}$ cross over.

When $v = 1$ then the contribution is zero by the Markov property. The main contribution will be from $v = 2$. Similar to [DU15], the contributions of terms with $v \geq 3$ can be bounded above by just considering the positive part, $\mathbb{E}\, L_{n,w}(\alpha)^2$. Let us first treat this case leaving $v = 2$ for later.

Case $v \geq 3$. Letting $\rho(\alpha)$ denote combinatorial factors, the contribution to $\mathbb{E}\, L_{n,w}(\alpha)^2$ from the terms with interlacement $v \geq 3$ is trivially bounded above by

$$\begin{aligned} I_n(w,\alpha) := \rho(\alpha) \sum_{k_1,\dots,k_{2\alpha}} \sum_{l_1,\dots,l_{2\alpha}} \Big\{ \mathbb{P}\Big[& S(k_1) = S(k_2) + A_2 w = \cdots = S(k_{2\alpha}) + A_{2\alpha} w; \\ & S(l_1) = S(l_2) + A'_2 w = \cdots = S(l_{2\alpha}) + A'_{2\alpha} w \Big] \\ = \rho(\alpha) \sum_{k_1,\dots,k_{2\alpha}} \sum_{l_1,\dots,l_{2\alpha}} \sum_x \mathbb{P}\Big[& S(k_1) = \cdots = S(k_{2\alpha}) + A_{2\alpha} w; \\ & S(l_1) = S(k_1) - x, S(l_1) = S(l_2) + A'_2 w = \cdots = S(l_{2\alpha}) + A'_{2\alpha} w \Big], \end{aligned}$$

where $A_i, A'_i \in \mathbb{Z}$ and may vary from line to line. Let $(j_1,\dots,j_{4\alpha})$ denote the combined sequence, allowing for matches. Changing variables

$$j_1 = m_0,\ j_2 = m_0 + m_1, \dots,\ j_{4\alpha} = m_0 + \cdots + m_{4\alpha-1},\ n = m_0 + \cdots + m_{4\alpha},$$

with $m_0,\dots,m_{4\alpha} \geq 0$, we get

$$\begin{aligned} I_n(w,\alpha) \leq \rho(\alpha) \sum_{m_0,\dots,m_{4\alpha-1}\geq 0} \sum_x \mathbb{P}\Big[& S(m_1) = S(m_1+m_2) + A_2 w + \delta_2 x = \\ & \cdots = S(m_1 + \cdots + m_{4\alpha}) + A_{4\alpha} w + \delta_{4\alpha} x \Big] \end{aligned}$$

where $\delta_i := \epsilon_i - \epsilon_{i+1} \in \{-1, 0, +1\}$, and $\boldsymbol{\epsilon}$ is defined as earlier. A simple application of the Markov property results in

$$I_n(w,\alpha) \leq \rho(\alpha) n \sum_{m_1,\dots,m_{4\alpha-1}\geq 0} \sum_x \prod_{k=1}^{4\alpha-1} \mathbb{P}[S(m_k) = (\delta_{k-1} - \delta_k)x + A_k w],$$

where the factor n resulted from the free index m_0. Notice that since v is the number of interlacements, exactly $u := 4\alpha - 1 - v$ of the δ's are 0 and thus by (17)

$$I_n(w,\alpha) \leq Cn\log(n)^{4\alpha-1-v} \sum_{j_1,\dots,j_v} \sum_x \prod_{t=1}^{v} \mathbb{P}[S(j_t) = \delta'_t x + A_t w] \tag{21}$$

where $\delta'_t \in \{-1,+1\}$. Letting

$$D_{n,v} := \sum_{j_1,\dots,j_v=0}^{n} \sum_x \prod_{k=1}^{v} \mathbb{P}[S(j_k) = \delta'_k x + A_k w],$$

notice that

$$D_{n,v} \leq D_{n,v-1} \sum_{j_v} \sup_y \mathbb{P}\left[S(j_v) = y\right] \leq C D_{n,v-1} \sum_{j_v=1}^{n} \frac{1}{j_v} \leq C\log(n) D_{n,v-1}.$$

Repeating we arrive at $D_{n,v} \le C\log(n)^{v-3}D_{n,3}$, and therefore

$$I_n(w,\alpha) \le Cn\log(n)^{4\alpha-4}D_{n,3}.$$

To complete our study of the $v \ge 3$ case we now treat the term $D_{n,3}$.

$$\begin{aligned} D_{n,3} &\le C\sum_{i\le j\le k}\sum_x \mathbb{P}[S(i)=\delta'_i x + A_i w]\times\mathbb{P}[S(j)=\delta'_j x + A_j w]\times\mathbb{P}[S(k)=\delta'_k x + A_k w] \\ &\le C\sum_{i\le j\le k}\Big(\sup_y \mathbb{P}[S(j)=y]\Big)\sup_y \mathbb{P}[S(i+k)=y], \end{aligned}$$

where $\tilde{S}_k$ denotes an independent copy of S_k. By symmetry and Lemma 7

$$\begin{aligned} D_{n,3} &\le C\sum_{0\le i\le j\le k\le n}\frac{1}{j}\frac{1}{i+k} \le C\sum_{m_1,m_2,m_3=0}^{n}\frac{1}{m_1+m_2}\frac{1}{2m_1+m_2+m_3} \\ &\le C\sum_{m_1,m_2=0}^{n}\frac{1}{m_1+m_2}\log\Big(1+\frac{n}{m_1+m_2}\Big) \le C\sum_{j=0}^{2n}\log\Big(1+\frac{n}{j}\Big) \\ &\le C\int_{x=1}^{2n}\log\Big(1+\frac{n}{x}\Big)\mathrm{d}x \le n\int_{1/k}^{n}\log(1+y)\frac{\mathrm{d}y}{y^2} \le Cn. \end{aligned}$$

Therefore $D_{n,3} = O(n)$ and thus the total contribution of the terms with $v \ge 3$ is $O(n^2\log(n)^{4\alpha-4})$.
Case $v = 2$. Letting $M_n(4\alpha)$ be defined as usual, we have for some q that $l_1, \dots l_{2\alpha} \in [k_q, k_{q+1}]$. Denoting by $J_n(w,\alpha)$ the contribution of a single term with $v = 2$

$$\begin{aligned} J_n(w,\alpha) = &\sum_{M_n(4\alpha)}\prod_{\substack{1\le k\le 4\alpha-1\\ k\ne q,q+2\alpha}}\mathbb{P}[S(m_k)=A_k w] \\ &\times\Big[\mathbb{P}\Big(S(m_q)+S(m_{q+2\alpha})=K_1 w\Big)-\mathbb{P}\Big(S(m_q)+\cdots+S(m_{q+2\alpha})=K_2 w\Big)\Big], \end{aligned} \tag{22}$$

where K_1, K_2 are integers determined by $\mathbf{k}, \mathbf{l}$ and their interlacement. By (7) it follows that

$$\begin{aligned} J_n(w,\alpha) &\le Cn\log(n)^{2\alpha-2}\sum_{p_0,\dots,p_{2\alpha}}\frac{1}{p_2\cdots p_{2\alpha}}\Big[\frac{1}{p_0+p_1}-\frac{1}{p_0+p_1+\cdots+p_{2\alpha}}\Big] \\ &= Cn\log(n)^{2\alpha-2}\sum_{p_0,\dots,p_{2\alpha}}\frac{p_2+\cdots+p_{2\alpha}}{p_2\cdots p_{2\alpha}(p_0+p_1)(p_0+p_1+\cdots+p_{2\alpha})} \\ &\le C\alpha n\log(n)^{2\alpha-2}\sum_{p_0,\dots,p_{2\alpha}}\frac{1}{p_3\cdots p_{2\alpha}(p_0+p_1)(p_0+p_1+p_2+\cdots+p_{2\alpha})} \\ &\le C\alpha n\log(n)^{2\alpha-2}\sum_{p_2,\dots,p_{2\alpha}}\sum_{j=0}^{2n}\frac{1}{p_3\cdots p_{2\alpha}(j+p_2+\cdots+p_{2\alpha})} \\ &\le C\alpha n\log(n)^{2\alpha-2}\log(n)^{2\alpha-3+1}\sum_{p_1,p_2=0}^{n}\frac{1}{p_1+p_2} \le C\alpha n^2\log(n)^{4\alpha-4}. \end{aligned}$$

Thus the total contribution of terms with interlacement index $v = 2$ is $O\Big(n^2\log(n)^{4\alpha-4}\Big)$.

To complete the proof of Proposition 12, we first use Chebyshev's inequality to prove convergence along subsequences $n = \lfloor \rho^k \rfloor$, for $0 < \rho < 1$. We can fill in the gaps following the standard trick, as in [Če]. $\square$

5. Proof of Theorem 2

Our proof follows closely the outline of the proof of [Aa] and [KT]. The main difference in our approach is that we are using the a.s. Fölner property of the range and that we substitute the role of the local times with Theorem 5. In the following we assume that the entropy of S is finite. The case of infinite entropy can be easily derived by the same method.

Fix a finite generator β for S, the existence of which is a consequence of Krieger's Finite Generator Theorem [Kr] for $d=1$ and [KW, DP] for $d=2$. Let $\alpha=\{[x_1]:x\in\Omega\}$ be the partition of Ω according to the first coordinate. The partition $\Upsilon:=\alpha\times\beta$ is a countable generating partition of $\Omega\times Y$ for T. Thus by Aaronson's Generator Theorem (Theorem 1), what we need to show is that

$$\frac{\log n}{n}\log\mathcal{K}_{\mathcal{B}_\Omega\times\mathcal{Y}}\left(\Upsilon,n,\epsilon\right)\xrightarrow{m}\pi h\left(\mathsf{S}\right)\cdot\begin{cases}\gamma, & d=1,\ \mathbf{A1}\\ 2\sqrt{\det\Sigma}, & d=2,\ \mathbf{A2},\mathbf{A2'}.\end{cases}$$

For $a_0,a_1,..,a_n\in\Upsilon$ we write

$$[a_0,a_1,\cdots,a_n]:=\cap_{j=0}^{n}T^{-j}a_j$$

and the $\bar{d}_n$ metric on $\bigvee_{j=0}^{n-1}T^{-j}\Upsilon$,

$$\bar{d}_n\left([a_0,a_1,\cdots,a_{n-1}],[a_0',a_1',\cdots,a_{n-1}]\right):=\frac{\#\left\{0\le j\le n-1:a_j\neq a_j'\right\}}{n}.$$

It is straightforward to check that for all $n\in\mathbb{N}$ and $(w,y)\in\Omega\times Y$,

$$\left(\bigvee_{j=0}^{n-1}T^{-j}\Upsilon\right)(w,y)=\left[w_0^{n-1}\right]\times\beta_{R_n(w)}(y).$$

where $\beta_{R_n(w)}(y):=\left(\bigvee_{l\in R_n(w)}\mathsf{S}_l^{-1}\beta\right)(y)$ and $R_n(w):=\left\{\sum_{j=1}^{l}w_j:\ 1\le l\le n\right\}$ is the range of the random walk up to time n. For $n\in\mathbb{N}$, define $\Pi_n:\Omega\to 2^{\bigvee_{j=0}^{n-1}T^{-j}P}$ by

$$\Pi_n(w):\ =\ \left\{a\in\left(\bigvee_{j=0}^{n-1}T^{-j}\Upsilon\right):\ m\left(a\left|\mathcal{B}_\Omega\times Y\right.\right)(w)>0\right\}.$$

These are the partition elements seen by w. The function

$$\Phi_{n,\epsilon}(x):\ =\ \min\left\{\#F:\ F\subset\Pi_n(x),\ m\left(\cup_{a\in F}a\left|\mathcal{B}_\Omega\times Y\right.\right)>1-\epsilon\right\},$$

is an upper bound for $\mathcal{K}_{\mathcal{B}_\Omega\times\mathcal{Y}}\left(\Upsilon,n,\epsilon\right)(x)$ since in the definition of $\Phi_{n,\epsilon}$ we are using all sequences in $\Pi_n(x)$ on their own and not grouping them into balls.

To get a lower bound, introduce

$$\mathcal{Q}_{n,\epsilon}(x):=\max\left\{\#\left\{z\in\Pi_n(x):\ \bar{d}_n(a,z)\le\epsilon\right\}:\ a\in\Pi_n(x)\right\}$$

to be the maximal cardinality of elements of $\Pi_n(x)$ at a $\bar{d}_n$ ball centred at some $a\in\Pi_n(x)$. It then follows that

$$\mathcal{K}_{\mathcal{B}_\Omega\times Y}\left(P,n,\epsilon\right)(x)\ge\frac{\Phi_{n,\epsilon}(x)}{\mathcal{Q}_{n,\epsilon}(x)}.$$

Therefore the proof is separated into two parts. Firstly we prove that

$$\frac{\log n}{n}\log\Phi_{n,\epsilon}\xrightarrow{m}\pi h\left(\mathsf{S}\right)\cdot\begin{cases}\gamma, & d=1,\ \mathbf{A1}\\ 2\sqrt{\det\Sigma}, & d=2,\ \mathbf{A2},\mathbf{A2'},\end{cases}\tag{23}$$

and the second part consists of showing that

$$\frac{\log n}{n}\log\mathcal{Q}_{n,\epsilon}(x)\xrightarrow{m}0.\tag{24}$$

We will deduce (23) from the following Shannon Mcmillan Breiman Theorem.

Lemma 14. *For $\mathbb{P}$ almost every $w\in\Omega$,*

$$-\frac{\log n}{n}\log\nu\left(\beta_{R_n(w)}(y)\right)\xrightarrow{m}\pi h(\mathsf{S})\cdot\begin{cases}\gamma, & d=1,\ \mathbf{A1}\\ 2\sqrt{\det\Sigma}, & d=2,\ \mathbf{A2},\mathbf{A2'},\ \text{as } n\to\infty.\end{cases}$$

Proof. Let $d \in 1, 2$. By Theorem 6, for $\mathbb{P}$ almost every w, the range $\{R_n(w)\}$ is a Fölner sequence for $\mathbb{Z}^d$. Whence by Kieffer's Shannon-McMillan-Breiman Theorem [Ki], for $\mathbb{P}$ a.e. w,

$$-\frac{1}{\#R_n(w)}\log\nu\left(\beta_{R_n(w)}(y)\right)\xrightarrow[n\to\infty]{\nu} h(\mathsf{S})$$

and thus by Fubini,

$$-\frac{1}{\#R_n(w)}\log\nu\left(\beta_{R_n(w)}(y)\right)\xrightarrow[n\to\infty]{m} h(\mathsf{S}).$$

Notice that $h(\mathsf{S},\beta) = h(\mathsf{S})$ since β is a generating partition. Since by [DE] and (16),

$$\frac{\log n}{n}\#R_n(w)\xrightarrow[n\to\infty]{a.s.}\pi\begin{cases}\gamma, & d=1,\ \mathbf{A1},\\ 2\sqrt{\det\Sigma}, & d=2,\ \mathbf{A2},\mathbf{A2'},\end{cases}$$

the conclusion of the lemma follows. □

To keep the notations shorter, write

$$\mathsf{b}_d(n) := \frac{\pi n}{\log(n)}\begin{cases}\gamma, & d=1,\ \mathbf{A1},\\ 2\sqrt{\det\Sigma}, & d=2,\ \mathbf{A2},\mathbf{A2'}.\end{cases}$$

Proof of (23). Let $\epsilon > 0$ and for $n \in \mathbb{N}, x \in \Omega$ let

$$H_{n,x,\epsilon} := \left\{y \in Y :\ \nu\left(\beta_{R_n(x)}\right)(y) = e^{-\mathsf{b}_d(n)h(\mathsf{S})(1\pm\epsilon)}\right\}$$

By Lemma 14, there exists N_ϵ such that for all $n > N_\epsilon$, $\exists G_{n,\epsilon} \in \mathcal{B}_\Omega$ so that $\mathbb{P}(G_{n,\epsilon}) > 1-\epsilon$ and for all $x \in G_{n,\epsilon}$,

$$\nu(H_{n,x,\epsilon}) > 1 - \frac{\epsilon}{2}. \tag{25}$$

For $x \in G_{n,\epsilon}$, set $F_{n,x,\epsilon} := \left\{\beta_{R_n(x)}(y) :\ y \in H_{n,x,\epsilon}\right\}$. Since

$$\min\{\log\nu(a) : a \in F_{n,x,\epsilon}\} > -\mathsf{b}_d(n)h(\mathsf{S})(1+\epsilon)$$

one has by a standard counting argument that for $x \in G_{n,\epsilon}$

$$\log\Phi_{n,\epsilon}(x) \le \log\#F_{n,x,\epsilon} \le \mathsf{b}_d(n)h(\mathsf{S})(1+\epsilon)$$

On the other hand, it follows from (25) that for small ϵ and $x \in G_{n,\epsilon}$, if $F \subset \Pi_n(x)$ with $m\left(\bigcup_{a\in F} a|\,\mathcal{B}_\Omega \times Y\right)(x) > 1-\epsilon$ then for large n

$$\#F \ge \frac{1-3\epsilon/2}{\max\{\log\nu(a) : a \in F_{n,x,\epsilon}\}} \ge \frac{e^{\mathsf{b}_d(n)h(\mathsf{S})(1-\epsilon)}}{2}$$

Thus for every $x \in G_{n,\epsilon}$ with n large,

$$\log\Phi_{n,\epsilon}(x) \ge \mathsf{b}_d(n)h(\mathsf{S})(1-\epsilon) + \log(1/2) \ge \mathsf{b}_d(n)h(\mathsf{S})(1-2\epsilon).$$

The conclusion follows since

$$m\left([\log\Phi_{n,\epsilon}(x) = \mathsf{b}_d(n)h(\mathsf{S})(1\pm 2\epsilon)]\right) \ge \mathbb{P}(G_{n,\epsilon}) \xrightarrow[n\to\infty,\epsilon\to 0]{} 1.$$ □

5.1. **Proof of Equation** (24)**.** Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$2H(3\delta/2) + 3\delta\log(\#\beta) < \varepsilon$$

where for $0 < p < 1$,

$$H(p) = -p\log_2(p) - (1-p)\log_2(1-p).$$

is the entropy appearing in the Stirling approximation for the binomial coefficients. It follows from Theorem 5 that there exists $\mathsf{c} > 0$ and sets $A_{\delta,n} \in \mathcal{B}_\Omega$ (for all large n) such that for every $w \in A_{\delta,n}$,

$$\frac{\#\{x \in R_n(w) :\ l(n,x)(w) > \mathsf{c}\log(n)\}}{\#R_n(w)} > 1-\delta,$$

and $\mathbb{P}(A_{\delta,n}) > 1-\delta$. Since $\#R_n(w) \sim \mathsf{b}_d(n)$ almost surely we can assume further that for all $w \in A_{\delta,n}$, $\#R_n(w) \lesssim 2\mathsf{b}_d(n)$.

Since $\Pi_n(w) \subset \left[x_0^{n-1}\right] \times \beta_{R_n(w)}$, we can define a map $\mathbf{z} : \Pi_n(w) \to \beta_{R_n(w)}$ by

$$a =: \left[x_0^{n-1}\right] \times \mathbf{z}(a)$$

For $z \in \beta_{R_n(w)}$ and $j \in R_n(w)$, denote by z_j the element of β such that $z \subset \mathsf{S}_j^{-1}\beta$.

Lemma 15. *For large $n \in \mathbb{N}$ and $w \in A_{\delta,n}$, if $a, a' \in \Pi_n(w)$ then*

$$\#\left\{j \in R_n(w) : \ \mathbf{z}(a)_j \neq \mathbf{z}\left(a'\right)_j\right\} \leq \mathsf{b}_d(n)\left(\frac{\bar{d}_n\left(a, a'\right)}{\hat{\mathsf{c}}} + 2\delta\right),$$

where $\hat{\mathsf{c}} := \mathsf{c} \cdot \mathsf{b}_d(n)\log(n)/n$.

Proof. Define

$$K_n(w) := \left\{j \in R_n(w) : \ \mathbf{z}(a)_j \neq \mathbf{z}\left(a'\right)_j\right\}$$

and

$$F_n(w) := \left\{j \in R_n(w) : l(n,x)(w) \geq \mathsf{c}\log(n)\right\}.$$

Then $K_n \subset (K_n \cap F_n) \cup F_n^c$ and therefore since $w \in A_{\delta,n}$,

$$\begin{aligned}
\#K_n(w) &\leq \#\left(K_n \cap F_n\right)(w) + \#F_n^c(w) \\
&\leq \#\left(K_n \cap F_n\right)(w) + \delta\#R_n(w) \\
&\lesssim \#\left(K_n \cap F_n\right)(w) + 2\delta\mathsf{b}_d(n)
\end{aligned}$$

Finally,

$$\begin{aligned}
\#\left(K_n \cap F_n\right) &\leq \frac{1}{\mathsf{c}\log(n)}\sum_{j \in F_n(w)} l(n,j)\mathbf{1}_{K_n(w)} \\
&\leq \frac{1}{\mathsf{c}\log(n)}\#\left\{0 \leq i \leq n-1 : \ \mathbf{z}(a)_{s_i(w)} \neq \mathbf{z}(a')_{s_i(w)}\right\} \\
&= \frac{n}{\mathsf{c}\log(n)}\bar{d}_n\left(a, a'\right).
\end{aligned}$$

The conclusion follows. □

Proof of (24). First we show that for n large enough so that $A_{\delta,n}$ is defined,

$$\max_{w \in A_{\delta,n}} \log \mathcal{Q}_{n,\hat{\mathsf{c}}\delta}(w) \leq \varepsilon a_d(n)$$

To see this first notice that by Lemma 15 for every $a \in \Pi_n(w)$,

$$\left\{a' \in \Pi_n(w) : \bar{d}_n\left(a, a'\right) \leq \hat{\mathsf{c}}\delta\right\} \subset \left\{\mathbf{z} \in \beta_{R_n(w)} : \ \#\left\{j \in R_n(w) : \mathbf{z}(a)_j \neq \mathbf{z}_j\right\} \leq 3\delta\mathsf{b}_d(n)\right\}.$$

Thus for $w \in A_{\delta,n}$, using the Stirling approximation for the Binomial and $\#R_n(w) \lesssim 2\mathsf{b}_d(n)$,

$$\begin{aligned}
\log \mathcal{Q}_{n,\hat{\mathsf{c}}\delta}(w) &\leq \log\left[\binom{\#R_n(w)}{3\delta\mathsf{b}_d(n)}(\#\beta)^{3\delta\mathsf{b}_d(n)}\right] \\
&\lesssim 3\mathsf{b}_d(n)\delta\log(\#\beta) + \log\binom{2\mathsf{b}_d(n)}{3\delta\mathsf{b}_d(n)} \\
&\sim \mathsf{b}_d(n)\left[3\delta\log(\#\beta) + 2H(3\delta/2)\right] \\
&\leq \varepsilon\mathsf{b}_d(n).
\end{aligned}$$

This shows that for large n,

$$\mathbb{P}\left(\log \mathcal{Q}_{n,\hat{\mathsf{c}}\delta} > 2\varepsilon\mathsf{b}_d(n)\right) \leq \mathbb{P}\left(A_{\delta,n}^c\right) \leq \delta$$

and thus we have finished the proof of (24). □

As was mentioned before, Theorem 2 follows from (23) and (24).

Appendix A. Proofs of auxiliary results

Proof of Lemma 7. We only prove the second statement, the first being simpler. For $d = 1$ and any $\epsilon > 0$ by strong aperiodicity for $|t| > \epsilon$ it is true that $|\phi(t)| < C(\epsilon) < 0$. Therefore

$$\begin{aligned}\Big|\mathbb{P}[S(m) = 0] - \mathbb{P}[S_m = w]\Big| &\le \int_{-\pi}^{\pi} |1 - \mathrm{e}^{\mathrm{i}tw}||\phi(t)|^m \mathrm{d}t \\ &\le \int_{|t|<\epsilon} |1 - \mathrm{e}^{\mathrm{i}tw}||\phi(t)|^m \mathrm{d}t + 4\pi C(\epsilon)^m,\end{aligned}$$

where the second term decays exponentially. For the first term we have, since $\phi(t) = 1 - \gamma|t| + o(|t|)$, for ϵ small enough and $|t| < \epsilon$,

$$|\phi(t)| \le |1 - \gamma|t|| + D(\epsilon)|t| \le 1 - \frac{\gamma}{2}|t|.$$

Therefore

$$\begin{aligned}\int_{|t|<\epsilon} |1 - \mathrm{e}^{\mathrm{i}tw}||\phi(t)|^m \mathrm{d}t &\le C \int_{|t|<\epsilon} |t||w|\Big(1 - \frac{\gamma}{2}|t|\Big)^m \mathrm{d}t \\ &= C|w| \int_{t=0}^{\epsilon} t\Big(1 - \frac{\gamma t}{2}\Big)^m \mathrm{d}t \\ &\le C|w| \int_{t=0}^{\epsilon} t \exp\big(-\frac{m\gamma t}{2}\big)\mathrm{d}t \le C\frac{|w|}{m^2}.\end{aligned}$$

We prove (7) for $d = 1$. By (6) it suffices to consider $w = 0$. For the moment fix a small $\epsilon > 0$. Then, by aperiodicity, for $|t| > \epsilon$, there exists $\rho(\epsilon) \in (0, 1)$, such that $|\phi(t)| < \rho(\epsilon)$. Thus

$$\begin{aligned}\mathbb{P}[S(n) = 0] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t)^n \mathrm{d}t = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \phi(t)^n \mathrm{d}t + O(\rho(\epsilon)^n) \\ &= \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} [1 - \gamma|t| + R(t)]^n \mathrm{d}t + O(\rho(\epsilon)^n) \\ &=: I(n, \epsilon) + O(\rho(\epsilon)^n).\end{aligned}$$

Since $R(t) = o(t)$, for $|t| < \epsilon$ we can find $C(\epsilon)$ such that $|R(t)| \le C(\epsilon)|t|$ and such that $C(\epsilon) \to 0$ as $\epsilon \to 0$.

Therefore letting $\gamma_1(\epsilon) := \gamma(1 + C(\epsilon))$

$$\begin{aligned}I(n, \epsilon) &\ge \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} [1 - \gamma|t| - C(\epsilon)|t|]^n \mathrm{d}t = \frac{1}{\pi} \int_0^{\epsilon} [1 - \gamma_1(\epsilon)t]^n \mathrm{d}t \\ &= \frac{1}{\pi\gamma_1(\epsilon)} \int_0^{\gamma_1(\epsilon)\epsilon} [1 - t]^n \mathrm{d}t = \frac{1}{\pi\gamma_1(\epsilon)} \Big\{\frac{1}{n + 1} - \frac{[1 - \gamma_1(\epsilon)\epsilon]^{n+1}}{n + 1}\Big\}.\end{aligned}$$

Since for $\epsilon > 0$ small enough we have $0 < 1 - \gamma_1(\epsilon)\epsilon < 1$ we compute

$$\liminf_{n\to\infty} n\,\mathbb{P}[S(n) = 0] \ge \frac{1}{\pi\gamma_1(\epsilon)}.$$

On the other hand we also have

$$I(n, \epsilon) \le \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} [1 - \gamma|t| + C(\epsilon)|t|]^n \mathrm{d}t = \frac{1}{\pi} \int_0^{\epsilon} [1 - \gamma_2(\epsilon)t]^n \mathrm{d}t$$

where $\gamma_2(\epsilon) = 1 - C(\epsilon)$. Thus

$$\begin{aligned}I(n, \epsilon) &\le \frac{1}{\pi} \int_0^{\epsilon} [1 - \gamma_2(\epsilon)t]^n \mathrm{d}t = \frac{1}{\pi\gamma_2(\epsilon)} \int_0^{\gamma_2(\epsilon)\epsilon} [1 - t]^n \mathrm{d}t \\ &= \frac{1}{\pi\gamma_2(\epsilon)} \Big\{\frac{1}{n + 1} - \frac{[1 - \gamma_2(\epsilon)\epsilon]^{n+1}}{n + 1}\Big\}.\end{aligned}$$

For $\epsilon > 0$ small enough we have that $1 - \gamma_2(\epsilon)\epsilon \in (0, 1)$, and therefore we obtain that

$$\limsup_{n\to\infty} n\,\mathbb{P}[S(n) = 0] \le \frac{1}{\pi\gamma_2(\epsilon)}.$$

Since $\epsilon > 0$ is can be arbitrarily small and $\gamma_1(\epsilon), \lim \gamma_2(\epsilon) \to \gamma$, (7) follows.

For $d = 2$ the proof is similar, using polar coordinates. □

Proof of Lemma 8. Let $\delta > 0$ be arbitrary but small. Then

$$\frac{1}{2\pi}\int_{t=-\pi}^{\pi}\frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda\phi(t)} = \frac{1}{2\pi}\int_{|t|\le\delta}\frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda\phi(t)} + \frac{1}{2\pi}\int_{\pi\ge|t|>\delta}\frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda\phi(t)}.$$

By strong aperiodicity for small enough $\delta > 0$ there exists a small positive constant $D(\delta)$ such that $|\phi(t)| < 1 - D(\delta)$ when $|t| > \delta$. Thus

$$\Big|\frac{1}{2\pi}\int_{\pi\ge|t|>\delta}\frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda\phi(t)}\Big| \le CD(\delta)^{-1},$$

for all $\lambda \le 1$. Also

$$\frac{1}{2\pi}\int_{|t|\le\delta}\frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda\phi(t)} = \frac{1}{2\pi}\int_{|t|\le\delta}\frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda(1-\gamma|t|)} + I(\lambda,\delta),$$

where a standard argument using **A1** and the strong aperiodicity shows that there exists $r(\delta) = o_\delta(1)$, as $\delta \to 0$ such that

$$|I(\lambda,\delta)| \le r(\delta)\log\Big(\frac{1}{1-\lambda}\Big).$$

Finally as $\lambda \uparrow 1$ it is easily seen that

$$\begin{aligned}\frac{1}{2\pi}\int_{|t|\le\delta}\frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda(1-\gamma|t|)} &\sim \frac{1}{\pi}\int_{t=0}^{\delta}\frac{\mathrm{d}t}{1-\lambda+\lambda\gamma t}\\ &= \frac{1}{\pi}\int_{t=0}^{\delta}\frac{\mathrm{d}t}{1-\lambda+\lambda\gamma t} \sim \frac{1}{\pi\gamma}\log\Big(\frac{1}{1-\lambda}\Big).\end{aligned}$$

Therefore as $\lambda \uparrow 1$

$$\begin{aligned}\frac{1}{2\pi}\int_{t=-\pi}^{\pi}\frac{\lambda\phi(t)\mathrm{d}t}{1-\lambda\phi(t)} &= \frac{1}{\pi\gamma}\log\Big(\frac{1}{1-\lambda}\Big)(1+O(r(\delta)))+O(1)\\ &\sim \frac{1}{\pi\gamma}\log\Big(\frac{1}{1-\lambda}\Big),\end{aligned}$$

since δ is arbitrarily small and $r(\delta) \to 0$ as $\delta \to 0$.

□

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